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Ethereal Stratum Theory: A Mathematical Field Model

Scalar Field Formulation, Dynamics, Attractor Structure, and Cosmological Coupling

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Abstract

The Ethereal Stratum is a cosmological scalar field φ proposed to permeate all of spacetime, serving as the dynamical substrate underlying gravitation and large-scale structure formation. The model is rigorously defined by a fully covariant action combining the Einstein–Hilbert term with a canonical scalar Lagrangian, and yields a well-posed Klein–Gordon equation of motion on a curved background. Coupling to the Einstein field equations is realized through the canonical stress-energy tensor, reproducing standard expressions for energy density, pressure, and the equation of state parameter across all dynamical regimes.

Beyond the minimal six-step core, this document develops the theory's dynamical structure in depth: slow-roll inflationary dynamics and the resulting observable spectral tilt and tensor-to-scalar ratio; three fully worked exact and near-exact solutions of the background equations (monomial, exponential, and quadratic potentials), each corresponding to a distinct cosmological epoch — inflationary slow roll, dark-energy tracking, and coherent matter-like oscillation; a k-essence generalisation of the kinetic term and its associated sound speed; and the leading-order fifth-force phenomenology that results if the Ethereal Stratum is allowed to couple conformally to Standard Model matter. Every new result is derived explicitly from the action of Section 3, so that the extended theory remains, like the core, entirely self-contained

and free of unexplained parameters beyond $V(\varphi)$, M_{Pl}, and (where invoked) a single coupling function $A(\varphi)$.

1. Introduction

The long-standing challenge of identifying a dynamical origin for gravitational structure in the cosmos has motivated numerous field-theoretic approaches to cosmology. Among these, scalar field models occupy a distinguished position: they are Lorentz-invariant, general-covariant, and admit a clean variational formulation that connects naturally to both quantum field theory and classical general relativity. The Ethereal Stratum Theory proposes that a single real scalar field φ, defined globally on spacetime, encodes the physical degrees of freedom of a continuous medium — the Ethereal Stratum — that carries energy density, pressure, and momentum, and which drives both the background cosmological expansion and the seeding of gravitational structure.

The theoretical foundation of this model rests on two pillars: the variational principle and general covariance. All dynamical equations are derived by extremising a diffeomorphism-invariant four-volume action with respect to the independent dynamical variables, namely the scalar field φ and the metric tensor $g_{\mu\nu}$. This ensures that the resulting field equations and stress-energy tensor are manifestly covariant and compatible with the equivalence principle. No background structure beyond the metric is assumed; the Ethereal Stratum is a genuine geometric field theory in the spirit of modern cosmological scalar field models.

Two independent variations of the same action generate the entire minimal theory. Varying with respect to φ, holding $g_{\mu\nu}$ fixed, yields the field's own equation of motion (Section 4). Varying with respect to $g_{\mu\nu}$, holding φ fixed, yields both the stress-energy tensor that sources gravity (Section 5) and the Einstein field equations themselves (Section 6). That a single action produces a fully closed system — field dynamics, sourced gravity, and their

mutual consistency — is the organising fact of the whole construction, and is verified explicitly in Section 5.2.

Sections 7–11 extend this minimal core along four directions that are logically independent of one another but share the same underlying action: the near-de Sitter (slow-roll) regime and its observable imprint on the primordial power spectrum (Section 7); three exactly or near-exactly solvable background histories illustrating the inflationary, dark-energy, and matter-like epochs the field can occupy (Section 8); a one-parameter-function generalisation of the kinetic term to a k-essence structure $P(X, \varphi)$ (Section 10); and a minimal conformal coupling of the Stratum to ordinary matter, together with the solar-system bound this coupling must satisfy (Section 11). Section 9 (structure formation) is unchanged in substance from the minimal theory but is retained in its original position for narrative continuity.

This document is entirely self-contained and proceeds from first principles. Section 2 introduces the scalar field and its covariant kinematic quantities. Section 3 presents the full action. Sections 4 and 5 derive the equation of motion and stress-energy density expressions, respectively. Section 6 establishes the coupling to the Einstein field equations and the Friedmann system. Section 7 develops slow-roll dynamics and inflationary observables. Section 8 presents three fully worked background solutions. Section 9 analyses cosmological perturbations and the structure formation chain. Section 10 generalises the kinetic sector to k-essence. Section 11 examines matter coupling and fifth-force constraints. Section 12 distils the minimal theory into a compact six-step formulation, and Section 13 summarises open directions. A notation reference table concludes the document.

2. Scalar Field Definition

The Ethereal Stratum is modelled by a real, smooth scalar field $\varphi = \varphi(x^\mu)$ defined on the four-dimensional pseudo-Riemannian spacetime manifold $(\mathcal{M}, g_{\mu\nu})$ equipped with the metric signature $(-, +, +, +)$. Being a scalar, φ is invariant under coordinate transformations and requires no additional gauge structure.

2.1 Covariant Kinetic Invariant

The covariant kinetic invariant X is defined as the contraction of the field gradient with itself, with an overall sign chosen so that $X > 0$ corresponds to a timelike gradient (canonical kinetic energy):

$$X \equiv -\tfrac{1}{2}\, g^{\mu\nu}\, \partial_\mu\varphi\, \partial_\nu\varphi \qquad (1)$$

For a spatially homogeneous field in a flat Friedmann–Lemaître–Robertson–Walker (FLRW) background, where spatial gradients vanish and the only dependence is on cosmic time t, one finds $X = \tfrac{1}{2}\dot{\varphi}^2$, so that X reduces to the familiar non-relativistic kinetic energy density.

2.2 Self-Interaction Potential

The function $V(\varphi)$ is a smooth, non-negative potential governing the self-interaction energy of the Ethereal Stratum field. The precise form of $V(\varphi)$ is model-dependent and determines the dynamical regime: a quadratic potential $V = \tfrac{1}{2}m^2\varphi^2$ produces oscillatory behaviour characteristic of coherent matter, while a shallower plateau or monomial form drives slow-roll inflation or late-time acceleration. Section 8 works through one representative potential of each type.

2.3 Physical Interpretation

The scalar φ encodes the local thermodynamic state of the Ethereal Stratum at each spacetime point. The kinetic invariant X measures the kinetic energy density of field excitations in a fully covariant sense, free of any preferred reference frame. The potential $V(\varphi)$ drives slow-roll or oscillatory behaviour depending on the cosmological epoch, providing a single unified framework that interpolates between inflationary, radiation-like, matter-like, and dark-energy-like dynamical regimes.

3. The Action

The dynamics of both spacetime geometry and the Ethereal Stratum field are encoded in the following covariant four-volume action, obtained by minimally coupling a canonical scalar to the Einstein–Hilbert gravitational term:

$$S = \int d^4x \sqrt{(-g)}\ [\, M_{Pl}^2/2 \cdot R + \mathcal{L}_\varphi \,] \qquad (2)$$

where the Lagrangian density of the Ethereal Stratum field is:

$$\mathcal{L}_\varphi = X - V(\varphi) = -\tfrac{1}{2}\, g^{\mu\nu}\, \partial_\mu\varphi\, \partial_\nu\varphi - V(\varphi) \qquad (3)$$

so that the complete action reads explicitly:

$$S = \int d^4x \sqrt{(-g)}\ [\, M_{Pl}^2/2 \cdot R - \tfrac{1}{2}\, g^{\mu\nu}\, \partial_\mu\varphi\, \partial_\nu\varphi - V(\varphi) \,] \qquad (4)$$

Here $M_{Pl} = (8\pi G)^{-1/2}$ is the reduced Planck mass, G is Newton's gravitational constant, R is the Ricci scalar curvature constructed from $g_{\mu\nu}$ and its derivatives, and $g = det(g_{\mu\nu})$ so that $\sqrt{(-g)}$ is the invariant spacetime volume element.

The first bracketed term is the Einstein–Hilbert action, whose variation with respect to $g_{\mu\nu}$ yields the Einstein tensor. The second and third terms together constitute the action of a canonical scalar field minimally coupled to gravity — no non-minimal curvature coupling $\xi R\varphi^2$ or kinetic mixing is introduced, keeping the model in its simplest and most predictive form. Section 10 revisits this choice and shows how it sits inside the larger k-essence family $\mathcal{L}_\varphi = P(X, \varphi)$ as the special case $P = X - V$.

Because the action depends on exactly two independent dynamical variables, φ and $g_{\mu\nu}$, exactly two variations exhaust the content of the minimal theory: $\delta S/\delta\varphi = 0$ (Section 4) and $\delta S/\delta g_{\mu\nu} = 0$ (Section 6, via the stress tensor constructed in Section 5).

4. Equation of Motion

Varying the action (4) with respect to φ and holding $g_{\mu\nu}$ fixed, the Lagrangian $\mathcal{L}_\varphi$ depends on φ only through φ itself and its first derivative $\partial_\mu\varphi$, so the field equation follows from the standard covariant Euler–Lagrange condition:

$$(1/\sqrt{(-g)})\,\partial_\mu\,[\,\sqrt{(-g)}\cdot\partial\mathcal{L}_\varphi/\partial(\partial_\mu\varphi)\,] - \partial\mathcal{L}_\varphi/\partial\varphi = 0$$

Since $\partial\mathcal{L}_\varphi/\partial(\partial_\mu\varphi) = -g^{\mu\nu}\partial_\nu\varphi$ and $\partial\mathcal{L}_\varphi/\partial\varphi = -V'(\varphi)$, substitution gives the covariant Klein–Gordon equation:

$$\Box\varphi - V'(\varphi) = 0 \qquad (5)$$

where the covariant d'Alembertian is:

$$\Box\varphi \equiv (1/\sqrt{(-g)})\,\partial_\mu\,[\,\sqrt{(-g)}\,g^{\mu\nu}\,\partial_\nu\varphi\,] \qquad (6)$$

and $V'(\varphi) = dV/d\varphi$. Note that this derivation never calls on the variation of $\sqrt{(-g)}$ itself — that identity belongs to the metric variation of Section 6, not to the field variation performed here, since $\sqrt{(-g)}$ does not depend on φ.

4.1 Homogeneous FLRW Reduction

For the spatially flat FLRW metric $ds^2 = -dt^2 + a^2(t)\,\delta_{ij}\,dx^i dx^j$, where $a(t)$ is the cosmological scale factor, $\sqrt{(-g)} = a^3$ and $g^{00} = -1$. For a spatially homogeneous field $\varphi = \varphi(t)$, only the $\mu = \nu = 0$ term of (6) survives:

$$\Box\varphi = (1/a^3)\,\partial_t\,[\,a^3(-1)\dot{\varphi}\,] = -(\ddot{\varphi} + 3H\dot{\varphi})$$

Substituting into (5) and multiplying through by -1 gives the reduced equation of motion:

$$\ddot{\varphi} + 3H\dot{\varphi} + V'(\varphi) = 0 \qquad (7)$$

where $H \equiv \dot{a}/a$ is the Hubble parameter. This is the damped nonlinear oscillator equation familiar from inflationary cosmology and quintessence models. The term $3H\dot{\varphi}$ is the Hubble friction term: the expansion of the universe damps the velocity of the field, slowing its roll down the potential $V(\varphi)$ and, in the slow-roll regime, sustaining a nearly constant potential energy density over many Hubble times.

5. Density Expressions

The stress-energy tensor of the Ethereal Stratum field is obtained by the standard variational definition:

$$T^{\varphi}_{\mu\nu} = -(2/\sqrt{(-g)}) \cdot \delta(\sqrt{(-g)}\,\mathcal{L}_{\varphi})/\delta g^{\mu\nu}$$

Since only the kinetic term of $\mathcal{L}_{\varphi}$ depends explicitly on the metric, $\delta\mathcal{L}_{\varphi} = -\tfrac{1}{2}\,\delta g^{\mu\nu}\,\partial_{\mu}\varphi\,\partial_{\nu}\varphi$, while the volume element contributes $\delta\sqrt{(-g)} = -\tfrac{1}{2}\sqrt{(-g)}\,g_{\mu\nu}\,\delta g^{\mu\nu}$. Collecting both pieces and using $\mathcal{L}_{\varphi} = X - V(\varphi)$:

$$\delta(\sqrt{(-g)}\,\mathcal{L}_{\varphi}) = \sqrt{(-g)}\,[\,-\tfrac{1}{2}\,\partial_{\mu}\varphi\,\partial_{\nu}\varphi - \tfrac{1}{2}\,g_{\mu\nu}\,\mathcal{L}_{\varphi}\,]\,\delta g^{\mu\nu}$$

so that $T^{\varphi}_{\mu\nu} = \partial_{\mu}\varphi\,\partial_{\nu}\varphi + g_{\mu\nu}\,\mathcal{L}_{\varphi}$. Expanding $\mathcal{L}_{\varphi}$ then yields the explicit stress-energy tensor:

$$T^{\varphi}_{\mu\nu} = \partial_{\mu}\varphi\,\partial_{\nu}\varphi - g_{\mu\nu}\,[\,\tfrac{1}{2}\,g^{\alpha\beta}\,\partial_{\alpha}\varphi\,\partial_{\beta}\varphi + V(\varphi)\,] \qquad (8)$$

In the homogeneous FLRW background, this takes the form of a perfect fluid with energy density ρ_{φ} and pressure p_{φ}:

$$\rho_\varphi = \tfrac{1}{2}\dot{\varphi}^2 + V(\varphi) \qquad (9)$$

$$p_\varphi = \tfrac{1}{2}\dot{\varphi}^2 - V(\varphi) \qquad (10)$$

The equation of state parameter:

$$w_\varphi = p_\varphi/\rho_\varphi = (\tfrac{1}{2}\dot{\varphi}^2 - V) / (\tfrac{1}{2}\dot{\varphi}^2 + V) \qquad (11)$$

5.1 Dynamical Limiting Cases

The equation of state parameter w_φ interpolates continuously between three physically distinct regimes.

Regime	Condition	Equation of State	Fluid Analogue
Kinetic domination	$\dot{\varphi}^2 \gg V(\varphi)$	$w_\varphi \to +1$	Stiff fluid
Slow-roll / potential domination	$V(\varphi) \gg \tfrac{1}{2}\dot{\varphi}^2$	$w_\varphi \to -1$	Cosmological constant / dark energy
Equal partition	$\dot{\varphi}^2 = 2V(\varphi)$	$w_\varphi = 0$	Pressureless (dust-like) matter

No additional fields or fluid components are introduced to obtain this range of behaviour: the same scalar φ, moving through different regions of its potential, reproduces the entire cosmologically relevant equation-of-state spectrum. Section 8 exhibits one concrete potential realising each row of this table.

5.2 Continuity Equation

The covariant conservation of the stress-energy tensor, $\nabla^\mu T^\varphi_{\mu\nu} = 0$, takes the standard perfect-fluid form in the FLRW background:

$$\dot{\rho}_\varphi + 3H(\rho_\varphi + p_\varphi) = 0 \qquad (12)$$

Substituting the expressions (9) and (10) gives $\dot{\rho}_\varphi + 3H\dot{\varphi}^2 = 0$, which is identically satisfied by the equation of motion (7): differentiating (9), $\dot{\rho}_\varphi = \dot{\varphi}\ddot{\varphi} + V'(\varphi)\dot{\varphi} = \dot{\varphi}[\ddot{\varphi} + V'(\varphi)] = \dot{\varphi}[-3H\dot{\varphi}] = -3H\dot{\varphi}^2$. This confirms the internal consistency of the system — $\nabla^\mu T^\varphi{}_{\mu\nu} = 0$ is not an independent postulate but a direct consequence of (7), exactly as required by the Bianchi identity $\nabla^\mu G_{\mu\nu} \equiv 0$ once (13) below is imposed.

6. Einstein Coupling

Varying the total action (4) with respect to the metric $g_{\mu\nu}$ combines the gravitational and matter contributions. For the Einstein–Hilbert term, the standard identity $\delta(\sqrt{(-g)}R)/\delta g^{\mu\nu} = \sqrt{(-g)}(R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu})$, up to a total-derivative (Gibbons–Hawking–York) boundary term, gives the Einstein tensor directly; for the field term, $\delta(\sqrt{(-g)}\mathcal{L}_\varphi)/\delta g^{\mu\nu} = -\tfrac{1}{2}\sqrt{(-g)}T^\varphi{}_{\mu\nu}$ by the definition used in Section 5. Setting the total variation to zero for arbitrary $\delta g^{\mu\nu}$ yields the Einstein field equations sourced entirely by the Ethereal Stratum:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = (1/M_{Pl}^2)\, T^\varphi{}_{\mu\nu} \qquad (13)$$

6.1 FLRW Friedmann Equations

Projecting equation (13) onto the time-time and spatial components in the FLRW background yields the two Friedmann equations. The 00-component gives the Hubble constraint:

$$H^2 = \rho_\varphi / (3M_{Pl}^2) = (1/3M_{Pl}^2)\,[\,\tfrac{1}{2}\dot{\varphi}^2 + V(\varphi)\,] \qquad (14)$$

The traceless combination of the ij-components, together with (14), gives the Raychaudhuri equation:

$$\dot{H} = -\varphi^2 / (2M_{Pl}^2) \qquad (15)$$

Together with the equation of motion (7), equations (14) and (15) form a closed autonomous dynamical system fully governing the cosmological evolution of the Ethereal Stratum background.

6.2 Deceleration Parameter and Cosmic Acceleration

The deceleration parameter:

$$q \equiv -\ddot{a}a/\dot{a}^2 = \tfrac{1}{2}(1 + 3w_\varphi) \qquad (16)$$

Cosmic acceleration ($\ddot{a} > 0$, $q < 0$) occurs when $w_\varphi < -1/3$, i.e., when the potential energy $V(\varphi)$ dominates over the kinetic energy $\tfrac{1}{2}\dot{\varphi}^2$.

7. Slow-Roll Dynamics and Inflationary Observables

When $V(\varphi)$ dominates strongly over the kinetic term, the field rolls slowly enough that $\ddot{\varphi}$ can be neglected in (7) and $\dot{H}$ can be neglected relative to H^2 in (14)–(15). This slow-roll regime is where the Ethereal Stratum can act as an inflaton, characterised by two small dimensionless parameters.

7.1 Slow-Roll Parameters

The Hubble slow-roll parameter is defined directly from (14)–(15) as:

$$\varepsilon_H \equiv -\dot{H}/H^2 = \dot{\varphi}^2 / (2M_{Pl}^2 H^2) \qquad (19)$$

so $\varepsilon_H \ll 1$ is precisely the statement that kinetic energy is subdominant. In the slow-roll approximation, (7) reduces to $3H\dot{\varphi} \approx -V'(\varphi)$ and (14) to $H^2 \approx V/(3M_{Pl}^2)$; substituting gives the potential slow-roll parameters:

$$\varepsilon_V \equiv (M_{Pl}^2/2)(V'/V)^2 \quad \eta_V \equiv M_{Pl}^2\, V''/V \tag{20}$$

with $\varepsilon_H \approx \varepsilon_V$ under the same approximation. Inflation proceeds as long as $\varepsilon_V \ll 1$ and $|\eta_V| \ll 1$, and ends when $\varepsilon_V = 1$.

The number of e-folds of expansion $N \equiv \int H\, dt$ remaining before the end of inflation:

$$N(\varphi) \approx (1/M_{Pl}^2) \int_{\varphi_{end}}^{\varphi} (V/V')\, d\varphi' \tag{21}$$

evaluated between the field value at horizon crossing and φ_{end} where $\varepsilon_V = 1$. Observationally relevant scales exit the Hubble radius at $N \approx 50\text{–}60$ e-folds before the end of inflation.

7.2 Curvature Perturbation Power Spectrum

During slow roll, the gauge-invariant comoving curvature perturbation $\zeta \equiv -H\delta\varphi/\dot{\varphi}$ inherits a power spectrum from Bunch–Davies vacuum fluctuations $\langle \delta\varphi^2 \rangle^{1/2} \approx H/2\pi$ at horizon crossing ($k = aH$):

$$\Delta_\zeta^2 \approx H^2 / (8\pi^2 M_{Pl}^2\, \varepsilon_V)\,|_{k=aH} \tag{22}$$

7.3 Spectral Index and Tensor-to-Scalar Ratio

Differentiating $\ln \Delta_\zeta^2$ with respect to $\ln k$ at fixed horizon-crossing condition:

$$n_s - 1 \equiv d\ln \Delta_\zeta^2 / d\ln k = 2\eta_V - 6\varepsilon_V \tag{23}$$

Tensor (gravitational-wave) perturbations give a tensor power spectrum $\Delta_h^2 \approx 2H^2/(\pi^2 M_{Pl}^2)$, hence a tensor-to-scalar ratio:

$$r \equiv \Delta_h^2 / \Delta_\zeta^2 = 16\varepsilon_V \tag{24}$$

Equations (20)–(24) constitute the complete slow-roll observable set: once $V(\varphi)$ is specified, $\varepsilon_V(\varphi)$ and $\eta_V(\varphi)$ fix $N(\varphi)$ via (21), which in turn fixes n_s and r at the field value corresponding to any chosen N. Section 8.1 evaluates this chain explicitly for a monomial potential.

8. Attractor Solutions and Worked Examples

The background system (7), (14), (15) is a coupled nonlinear dynamical system with no closed-form solution for generic $V(\varphi)$. Three choices of potential admit exact or controlled approximate solutions, and each corresponds to one row of Table 5.1.

8.1 Monomial Potential: Slow-Roll Inflation

Take $V(\varphi) = (\lambda/n)\varphi^n$ for constant $\lambda > 0$, integer $n \geq 1$, and $\varphi > 0$. Then $\varepsilon_V = n^2 M_{Pl}^2/(2\varphi^2)$ and $\eta_V = n(n-1)M_{Pl}^2/\varphi^2$. Slow roll ends at $\varphi_{end} = nM_{Pl}/\sqrt{2}$. Carrying out integral (21) gives $N(\varphi) \approx \varphi^2/(2nM_{Pl}^2)$ for $\varphi \gg \varphi_{end}$, so that $\varphi_N^2 \approx 2nNM_{Pl}^2$ at N e-folds before the end. The observables at horizon crossing as functions of n and N:

$$\varepsilon_V = n/(4N) \quad n_s - 1 = -(n+2)/(2N) \quad r = \frac{4n}{N} \tag{25}$$

For the case $n = 2$ ($V = \tfrac{1}{2}m^2\varphi^2$) and $N = 60$: $n_s \approx 0.967$, $r \approx 0.133$ — the classic quadratic chaotic-inflation prediction. Current bounds sit roughly a factor of 3–4 below this value of r, which disfavours the pure quadratic monomial as the inflationary sector on its own; smaller effective n, or a plateau potential, brings r down while keeping n_s close to the observed value. The monomial case is retained as a worked calibration of the machinery of Section 7 rather than the theory's final prediction for the inflationary epoch.

8.2 Exponential Potential: The Quintessence Tracking Attractor

Take $V(\varphi) = V_0 \exp(-\lambda\varphi/M_{Pl})$ for constant $\lambda > 0$, and consider the field-dominated universe. Seek a power-law attractor $a(t) \propto t^p$. Then $H = p/t$ and $\dot{H} = -p/t^2$, so (15) gives $\dot{\varphi}$

$= M_{Pl}\sqrt{(2p)}/t$, integrating to $\varphi(t) = M_{Pl}\sqrt{(2p)}\ ln\ t + const$. Substituting into $V(\varphi)$ and matching powers of t forces $\lambda\sqrt{(2p)} = 2$, fixing the attractor exponent:

$$p = 2/\lambda^2 \qquad (26)$$

Note that p is a pure number — λ is dimensionless (since φ/M_{Pl} is dimensionless inside the exponent), and M_{Pl} cancels completely out of the exponent-matching condition. Hence $a(t) \propto t^{2/\lambda^2}$. Using the FLRW identity $1 + w = 2/(3p)$ for any power-law attractor, the field's equation of state locks onto:

$$w_\varphi = -1 + \lambda^2/3 \qquad (27)$$

independent of initial conditions — the defining property of a tracker (attractor) solution. For $\lambda^2 < 2$ this attractor already satisfies the acceleration condition $w_\varphi < -1/3$, so the exponential potential is the minimal Ethereal Stratum realisation of a dynamical dark-energy attractor whose late-time equation of state is fixed by a single parameter λ.

8.3 Quadratic Potential: Coherent Oscillation as Matter

Take $V(\varphi) = \frac{1}{2}m^2\varphi^2$ in the regime $m \gg H$. Writing $\varphi(t) \approx A(t)cos(mt)$ with $A(t)$ slowly varying compared to $1/m$, equation (7) gives the envelope equation $\dot{A}/A \approx -3H/2$, so:

$$A(t) \propto a(t)^{-3/2} \qquad (28)$$

Over one oscillation period, $\langle\varphi^2\rangle = \langle m^2\varphi^2\rangle = 2\langle V\rangle$ by the virial theorem, so from (9)–(11) the period-averaged equation of state is:

$$\langle w_\varphi\rangle = \langle \tfrac{1}{2}\varphi^2 - V\rangle \,/\, \langle \tfrac{1}{2}\varphi^2 + V\rangle = 0 \qquad (29)$$

reproducing the equal partition row of Table 5.1 exactly, and the averaged energy density $\rho_\varphi = \tfrac{1}{2}m^2A^2 \propto a^{-3}$ by (28) — the same dilution law as pressureless matter. A coherently oscillating Ethereal Stratum field in a steep quadratic potential is therefore indistinguishable, at the level of the background expansion, from a pressureless matter fluid.

9. Structure Formation Chain

The homogeneous background solution describes a smooth, isotropic Ethereal Stratum. Small perturbations seed the gravitational inhomogeneities that give rise to all observed large-scale structure.

9.1 Perturbation Decomposition

The field and metric are decomposed into background and first-order perturbations:

$$\varphi(t,\boldsymbol{x}) = \varphi_0(t) + \delta\varphi(t,\boldsymbol{x}) \quad g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} \qquad (17)$$

where $\varphi_0(t)$ is the homogeneous background and $\delta\varphi(t,\boldsymbol{x})$ is a small spatially inhomogeneous fluctuation; $\bar{g}_{\mu\nu}$ is the background FLRW metric and $\delta g_{\mu\nu}$ its linear perturbation, decomposed into scalar potentials Φ and Ψ in the conformal Newtonian gauge.

9.2 Perturbed Equation of Motion

In the conformal Newtonian gauge and the sub-Hubble limit ($k \gg aH$):

$$\delta\ddot{\varphi} + 3H\delta\dot{\varphi} + [\,k^2/a^2 + V''(\varphi_0)\,]\,\delta\varphi = 0 \qquad (18)$$

where $V'' = d^2V/d\varphi^2$ acts as an effective mass squared for the perturbation. Equation (18) is a damped oscillator with squared frequency $\omega^2_k = k^2/a^2 + V''(\varphi_0)$. For $V''(\varphi_0) > 0$, $\omega^2_k > 0$ for every k and $\delta\varphi$ has no growing mode at any scale. A genuine instability occurs only where $V''(\varphi_0) < 0$ (a hilltop or inflection), in which case modes with k below the effective

Jeans wavenumber $k_J^2 \equiv -a^2 V''(\varphi_0)/H^2$ (comoving Jeans length $\lambda_J = 2\pi/k_J$) grow tachyonically while short-wavelength modes $k > k_J$ are stabilised by gradient pressure.

This tachyonic mechanism is not the route by which the Ethereal Stratum seeds the matter power spectrum in the chain below: there, it is the frozen super-horizon $\delta\varphi$ that sources metric potentials Φ, Ψ on re-entry (step 3), and it is the resulting matter overdensity δ_m — governed by its own gravitational growth equation, not by (18) — that grows. Section 10.1 generalises the k^2/a^2 coefficient of (18) once the kinetic term is no longer canonical.

9.3 Six-Step Structure Formation Chain

1. **Quantum fluctuations seed $\delta\varphi$.** During an early high-energy epoch, quantum vacuum fluctuations of the Ethereal Stratum field generate a nearly scale-invariant spectrum of perturbations $\delta\varphi$ with power $\langle|\delta\varphi_k|^2\rangle \propto H^2/k^3$, quantified explicitly by (22)–(23).
2. **Inflation stretches modes beyond the Hubble radius.** Exponential expansion carries perturbation modes from sub-Hubble ($k \gg aH$) to super-Hubble ($k < aH$) scales, freezing them as classical, correlated inhomogeneities.
3. **Re-entry sources metric perturbations.** After inflation, modes re-enter the Hubble radius and $\delta\varphi$ sources gravitational potential perturbations Φ and Ψ through the perturbed Einstein equations (13).
4. **Gravitational potential wells deepen.** Growing metric perturbations deepen potential wells, driving growth of the fractional matter overdensity $\delta_m = \delta\rho_m/\rho_m$ and sourcing the matter power spectrum.
5. **Baryons fall into potential wells.** Baryonic matter accretes into the deepened potential wells, forming the first proto-galactic halos, which evolve through hierarchical merging into galaxies and galaxy clusters.
6. **Late-time acceleration suppresses growth.** The background field $\varphi_0(t)$ evolves toward potential-dominated slow-roll — realised explicitly by the tracking

attractor of Section 8.2 — driving $w_\varphi \to -1$, accelerating cosmic expansion, and producing the characteristic turnover in the large-scale matter power spectrum.

10. Generalized Kinetic Term: k-essence Extension

Section 3 fixed $\mathcal{L}_\varphi = X - V(\varphi)$, a purely canonical choice. Nothing in the variational structure of Sections 4–6 required this particular form; it is the simplest member of the wider k-essence family:

$$\mathcal{L}_\varphi = P(X, \varphi) \tag{30}$$

with the canonical theory recovered as the special case $P(X, \varphi) = X - V(\varphi)$.

10.1 Generalised Stress Tensor and Fluid Variables

Repeating the metric variation of Section 5 with $\mathcal{L}_\varphi = P(X, \varphi)$, noting that X itself depends on $g_{\mu\nu}$ through (1) so that $\delta X = -\tfrac{1}{2}\partial_\mu\varphi\partial_\nu\varphi\,\delta g^{\mu\nu}$, the chain rule gives the generalised stress tensor:

$$T^{\varphi}_{\mu\nu} = P_{,X}\,\partial_\mu\varphi\,\partial_\nu\varphi + g_{\mu\nu}\,P \tag{31}$$

where $P_{,X} \equiv \partial P/\partial X$. Equation (8) is recovered on setting $P_{,X} = 1$. In the homogeneous FLRW background this takes perfect-fluid form with:

$$\rho = 2X\,P_{,X} - P \quad p = P \tag{32}$$

so that $\rho = \dot{\varphi}^2\,P_{,X} - P$ reduces to (9) once more when $P_{,X} = 1$ and $P = X - V$.

10.2 Effective Sound Speed and the Perturbation Equation

Because P can depend nonlinearly on X, perturbations $\delta\varphi$ no longer propagate at unit sound speed. Expanding the quadratic action for $\delta\varphi$ about the background $X_0 = \tfrac{1}{2}\dot{\varphi}_0^2$ shows that the gradient term k^2/a^2 in (18) is rescaled by an effective sound speed:

$$c_s^2 = P_{,X} \,/\, (P_{,X} + 2X\,P_{,XX}) \tag{33}$$

so that the perturbed equation of motion generalises to $\delta\ddot{\varphi} + 3H\delta\dot{\varphi} + [c_s^2\,k^2/a^2 + V''_{\rm eff}]\,\delta\varphi = 0$, with $c_s^2 = 1$ identically for the canonical theory of Section 9. A sound speed $c_s^2 < 1$ shifts the effective Jeans wavenumber to $k_J^2 = c_s^2\,a^2 V''/H^2$, allowing structure to form on somewhat smaller comoving scales than in the canonical case.

11. Coupling to Matter and Fifth-Force Constraints

The minimal theory of Sections 2–6 couples the Ethereal Stratum to Standard Model matter only gravitationally. A minimal and commonly considered extension allows matter fields to propagate on a conformally related metric $\tilde{g}_{\mu\nu} = A^2(\varphi) g_{\mu\nu}$, so that φ mediates an additional force between matter particles beyond ordinary gravity.

11.1 Conformal Coupling and the Fifth-Force Parameter

With matter minimally coupled to $\tilde{g}_{\mu\nu} = A^2(\varphi) g_{\mu\nu}$ rather than to $g_{\mu\nu}$ directly, variation of the matter action with respect to φ produces a source term proportional to the trace of the matter stress tensor times:

$$\beta(\varphi) \equiv M_{\rm Pl} \cdot d(\ln A)/d\varphi \tag{34}$$

the dimensionless coupling strength of the fifth force mediated by φ. For β constant and of order unity, the extra force between two test masses has the same $1/r^2$ falloff as gravity and a strength β^2 times Newton's law — entirely unsuppressed unless β is small or a screening mechanism hides the coupling in high-density environments.

11.2 Solar-System Bound

In the weak-field, quasi-static limit, the conformal coupling (34) shifts the post-Newtonian parameter γ ($\gamma = 1$ in general relativity) away from unity by:

$$|\gamma - 1| \approx 2\beta^2 / (1 + \beta^2) \qquad (35)$$

The Cassini spacecraft time-delay experiment constrains $|\gamma - 1| = (2.1 \pm 2.3) \times 10^{-5}$ (Bertotti, Iess & Tortora, 2003), so any unscreened, order-unity-range Ethereal Stratum coupling must satisfy $\beta^2 \lesssim 10^{-5}$ near the Sun. This is not a difficulty for the minimal theory of Sections 2–6, which sets $\beta \equiv 0$ identically; it becomes a genuine constraint only if Section 11.1's extension is adopted.

12. Compact Six-Step Formulation

Compact Six-Step Formulation

—

Ethereal Stratum Theory

Step 1 — Define the Field

$\varphi : \mathcal{M} \to \mathbb{R} \quad X = -\tfrac{1}{2}(\partial\varphi)^2$

A real scalar field on the spacetime manifold with covariant kinetic invariant X. This defines the fundamental degree of freedom of the Ethereal Stratum.

Step 2 — Write the Action

$S = \int d^4x \sqrt{(-g)}\, [\, M_{Pl}^2/2 \cdot R + X - V(\varphi)\,]$

Einstein–Hilbert gravity plus a canonical scalar field, minimally coupled. The action is diffeomorphism-invariant by construction.

Step 3 — Vary to Get the Equation of Motion

$$\Box\varphi = V'(\varphi) \quad \rightarrow \quad \ddot{\varphi} + 3H\dot{\varphi} + V' = 0$$

The covariant Klein–Gordon equation, reducing to a damped oscillator with Hubble friction in the FLRW background.

Step 4 — Read Off the Density Expressions

$$\rho_\varphi = \tfrac{1}{2}\dot{\varphi}^2 + V \quad p_\varphi = \tfrac{1}{2}\dot{\varphi}^2 - V \quad w_\varphi = p_\varphi/\rho_\varphi$$

Kinetic plus potential decomposition; the equation of state parameter w tracks the dynamical regime from stiff (w = +1) through matter-like (w = 0) to dark-energy-like (w = –1).

Step 5 — Couple to the Einstein Equations

$$H^2 = \rho_\varphi / (3M_{Pl}^2) \quad \dot{H} = -\dot{\varphi}^2 / (2M_{Pl}^2)$$

The Ethereal Stratum drives cosmic expansion. Acceleration occurs when the potential energy dominates, yielding w < –1/3.

Step 6 — Perturb and Form Structure

$$\delta\ddot{\varphi} + 3H\delta\dot{\varphi} + (k^2/a^2 + V'')\delta\varphi = 0 \quad \rightarrow \quad \delta\rho_m \text{ grows} \quad \rightarrow \quad \text{galaxies}$$

Fluctuations in the Ethereal Stratum seed all observed large-scale structure through the full hierarchy: quantum seed → inflation → Hubble re-entry → gravitational collapse → galaxies and clusters.

13. Summary and Outlook

The Ethereal Stratum Theory provides a unified scalar field framework for encoding the dynamical substrate of spacetime. Beginning from a single real scalar field φ defined on the pseudo-Riemannian manifold of general relativity, the theory constructs a fully covariant action, derives all dynamical equations from the variational principle, and establishes the complete coupling between the field and the gravitational sector. The energy density and

pressure expressions follow directly from the canonical stress-energy tensor, and the equation of state parameter w_φ interpolates naturally between all cosmologically relevant fluid regimes without requiring additional fields or ad hoc parameterisations.

The six-step derivation chain presented in Section 12 demonstrates that the path from first-principles field definition to the prediction of observable large-scale structure is compact, logically sequential, and entirely self-contained. Each step follows from the previous by a well-defined mathematical operation — field definition, action construction, variation, stress-energy projection, gravitational coupling, and linearised perturbation analysis — with no free parameters beyond the potential function $V(\varphi)$ and the reduced Planck mass M_{Pl}. As shown in Section 5.2, the continuity equation $\nabla^\mu T^\varphi_{\mu\nu} = 0$ is not an independent input but follows automatically from the equation of motion, so the closure of the system is a theorem of the construction rather than an additional assumption.

Sections 7–11 show that this same minimal core, without adding any field beyond φ, already supports a slow-roll inflationary phase with computable observables (25), an exact dark-energy tracking attractor with a single free parameter (26)–(27), a matter-mimicking coherent-oscillation phase (28)–(29), a controlled generalisation of the kinetic sector with its own sound speed (33), and a well-defined, falsifiable fifth-force phenomenology (34)–(35) should the field be allowed to couple directly to matter. That a single scalar and a single potential function can be walked through inflation, structure formation, and dark energy in turn — each transition governed by nothing more than where φ sits on $V(\varphi)$ — is the theory's central economy claim.

Several important extensions and open directions remain. First, the k-essence generalisation of Section 10 could be specialised to a concrete $P(X, \varphi)$ motivated by a UV completion, allowing the sound speed (33) to be predicted rather than fitted. Second, a global potential $V(\varphi)$ that smoothly interpolates between the monomial, plateau, exponential, and quadratic shapes of Section 8 — so that a single field realises inflation, quintessence, and oscillatory matter-like behaviour as different epochs of one continuous

roll — would elevate Section 8 from three separate worked examples to a single unified history. Third, if the conformal coupling of Section 11 is retained, an explicit screening mechanism (chameleon or symmetron) should be built into $A(\varphi)$ and checked against laboratory fifth-force searches as well as the solar-system bound (35). Fourth, precision observational constraints from CMB anisotropy spectra, baryon acoustic oscillations, and weak gravitational lensing surveys (Euclid, CMB-S4, DESI) can be used to constrain the shape and amplitude of $V(\varphi)$ directly, using the spectral observables of Section 7.3 and the tracking equation of state (27) as concrete targets. Finally, a quantum field theoretic grounding of the potential — whether through spontaneous symmetry breaking, an axion-like shift symmetry, or an asymptotically safe gravitational fixed point — remains the deepest open question for elevating the Ethereal Stratum from a phenomenological framework to a fundamental theory.

Appendix A. Notation Reference

Symbol	Description
φ	*Ethereal Stratum scalar field; a real, smooth function on spacetime*
$g_{\mu\nu}$	*Spacetime metric tensor with signature (−, +, +, +)*
X	*Covariant kinetic invariant:* $X = -\frac{1}{2} g^{\mu\nu} \partial_\mu\varphi \partial_\nu\varphi$
$V(\varphi)$	*Self-interaction potential of the field; smooth and non-negative*
S	*Total covariant four-volume action (gravitational + field)*
M_{Pl}	*Reduced Planck mass:* $M_{Pl} = (8\pi G)^{-1/2} \approx 2.435 \times 10^{18}$ *GeV*
R	*Ricci scalar curvature, constructed from* $g_{\mu\nu}$ *and its second derivatives*
$\Box$	*Covariant d'Alembertian:* $\Box\varphi = (1/\sqrt{(-g)})\, \partial_\mu[\sqrt{(-g)}\, g^{\mu\nu} \partial_\nu\varphi]$
H	*Hubble parameter:* $H = \dot{a}/a$*; rate of expansion of the universe*
$a(t)$	*Cosmological scale factor in the FLRW metric*
ρ_φ	*Energy density of the Ethereal Stratum:* $\frac{1}{2}\dot{\varphi}^2 + V(\varphi)$
p_φ	*Pressure of the Ethereal Stratum:* $\frac{1}{2}\dot{\varphi}^2 - V(\varphi)$
w_φ	*Equation of state parameter:* $w_\varphi = p_\varphi/\rho_\varphi \in [-1, +1]$

Symbol	Description
$G_{\mu\nu}$	*Einstein tensor:* $G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu}$
$T_{\mu\nu}$	*Stress-energy tensor of the Ethereal Stratum field*
$\delta\varphi$	*First-order scalar field perturbation about the homogeneous background*
k	*Comoving wavenumber (Fourier conjugate to comoving position)*
q	*Deceleration parameter:* $q = -\ddot{a}a/\dot{a}^2 = \tfrac{1}{2}(1 + 3w_\varphi)$
ε_H, ε_V	*Hubble / potential slow-roll parameters:* $\varepsilon_H = -\dot{H}/H^2$, $\varepsilon_V = (M_{Pl}^2/2)(V'/V)^2$
η_V	*Second potential slow-roll parameter:* $\eta_V = M_{Pl}^2\, V''/V$
N	*Number of e-folds before the end of inflation:* $N = \int H\, dt$
Δ^2_ζ	*Dimensionless power spectrum of the comoving curvature perturbation* ζ
n_s	*Scalar spectral index:* $n_s - 1 = d \ln \Delta^2_\zeta / d \ln k$
r	*Tensor-to-scalar ratio:* $r = \Delta^2_h / \Delta^2_\zeta$
λ	*Exponential-potential slope parameter,* $V = V_0\, e^{-\lambda\varphi/M_{Pl}}$ *(Section 8.2)*
n	*Monomial-potential power,* $V = (\lambda/n)\varphi^n$ *(Section 8.1)*
$P(X,\varphi)$	*Generalised k-essence Lagrangian,* $\mathcal{L}_\varphi = P(X,\varphi)$*; canonical case* $P = X - V$
c^2_s	*Effective sound speed:* $c^2_s = P_{,X}/(P_{,X} + 2X\, P_{,XX})$
$A(\varphi)$	*Conformal coupling factor relating the matter-frame metric* $\tilde{g}_{\mu\nu} = A^2(\varphi) g_{\mu\nu}$
β	*Fifth-force coupling strength:* $\beta = M_{Pl}\, d(\ln A)/d\varphi$
γ	*Post-Newtonian (Eddington) parameter;* $\gamma = 1$ *in general relativity*

Appendix B. Derivation Notes and Verification Log

This appendix records the independent hand-verification of every non-trivial derivation in the main text, together with the corrections that verification produced. Each item re-derives the boxed result from the equation immediately preceding it in the main text.

B.1 Method

Every numbered equation was re-derived independently from the action (4) (or, for Sections 7–11, from (4) plus the stated approximation or generalisation), checked for sign

consistency under the (−,+,+,+) signature, and where possible cross-checked against a known limit or an independent literature result.

B.2 Core Theory (Sections 4–6)

FLRW d'Alembertian, eqs. (6)–(7). With $\sqrt{(-g)} = a^3$ and $g^{00} = -1$ for a homogeneous $\varphi(t)$, only the $\mu = \nu = 0$ term of (6) contributes:

> $\Box\varphi = (1/a^3)\,\partial_t[\,a^3(-1)\dot{\varphi}\,] = -(1/a^3)(3a^2\dot{a}\dot{\varphi} + a^3\ddot{\varphi}) = -(\ddot{\varphi} + 3H\dot{\varphi})$
>
> Substituting into (5): $\Box\varphi - V' = 0 \Rightarrow \ddot{\varphi} + 3H\dot{\varphi} + V' = 0$, matching (7) exactly.

Stress tensor, eq. (8). Only the kinetic piece of $\mathcal{L}_\varphi = X - V$ depends on $g_{\mu\nu}$ explicitly: $\delta\mathcal{L}_\varphi = -\tfrac{1}{2}\partial_\mu\varphi\partial_\nu\varphi\ \delta g^{\mu\nu}$, and $\delta\sqrt{(-g)} = -\tfrac{1}{2}\sqrt{(-g)}\ g_{\mu\nu}\ \delta g^{\mu\nu}$. Hence $\delta(\sqrt{(-g)}\mathcal{L}_\varphi)/\delta g^{\mu\nu} = \sqrt{(-g)}[-\tfrac{1}{2}\partial_\mu\varphi\partial_\nu\varphi - \tfrac{1}{2}g_{\mu\nu}\,\mathcal{L}_\varphi]$, and $T_{\mu\nu} = -2/\sqrt{(-g)} \times (that) = \partial_\mu\varphi\partial_\nu\varphi + g_{\mu\nu}\,\mathcal{L}_\varphi$. Expanding $\mathcal{L}_\varphi = -\tfrac{1}{2}g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi - V$ gives $T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - g_{\mu\nu}[\tfrac{1}{2}g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi + V]$, reproducing (8) term for term.

Continuity equation, eq. (12). Differentiating (9): $\dot{\rho}_\varphi = \dot{\varphi}\ddot{\varphi} + V'\dot{\varphi} = \dot{\varphi}(\ddot{\varphi} + V') = \dot{\varphi}(-3H\dot{\varphi}) = -3H\dot{\varphi}^2$, which is exactly (12) once $\rho_\varphi + p_\varphi = \dot{\varphi}^2$ (from (9)–(10)) is substituted. This is an independent check confirming that the equation of motion (7) alone is sufficient to reproduce conservation of $T^{\varphi}{}_{\mu\nu}$.

Friedmann equations, eqs. (14)–(15). Standard FLRW Einstein-tensor components give $G_{00} = 3H^2$ and, combined with the trace, $\ddot{a}/a = -(\rho+3p)/(6M_{Pl}^2)$. Equation (13) then gives $H^2 = \rho_\varphi/(3M_{Pl}^2)$ directly (eq. 14). For (15): $\dot{H} = \ddot{a}/a - H^2 = -(\rho_\varphi + 3p_\varphi)/(6M_{Pl}^2) - \rho_\varphi/(3M_{Pl}^2) = -(\rho_\varphi + p_\varphi)/(2M_{Pl}^2)$; substituting $\rho_\varphi + p_\varphi = \dot{\varphi}^2$ gives $\dot{H} = -\dot{\varphi}^2/(2M_{Pl}^2)$, matching (15).

Deceleration parameter, eq. (16). From $\ddot{a}/a = -H^2(1+3w_\varphi)/2$, $q \equiv -(\ddot{a}/a)/H^2 = (1+3w_\varphi)/2$, matching (16).

B.3 Slow-Roll Sector (Section 7)

ε_V from ε_H, eqs. (19)–(20). Under $3H\dot{\varphi} \approx -V'$ and $H^2 \approx V/(3M_{Pl}^2)$:

> $\varepsilon_H = \dot{\varphi}^2/(2M_{Pl}^2H^2) = [V'^2/(9H^2)]/(2M_{Pl}^2H^2) = V'^2/(18M_{Pl}^2H^4)$
>
> Substituting $H^4 \approx V^2/(9M_{Pl}^4)$:
>
> $\varepsilon_H \approx V'^2 \cdot 9M_{Pl}^4/(18M_{Pl}^2 \cdot V^2) = M_{Pl}^2\, V'^2/(2V^2) = (M_{Pl}^2/2)(V'/V)^2 = \varepsilon_V$
>
> confirming (20) follows from (19) under the stated approximation.

Tensor-to-scalar ratio, eq. (24). $r \equiv \Delta_h^2/\Delta_\zeta^2 = [2H^2/(\pi^2M_{Pl}^2)] \,/\, [H^2/(8\pi^2M_{Pl}^2\,\varepsilon_V)] = 16\varepsilon_V$, an algebraic identity verified by direct substitution.

B.4 Worked Examples (Section 8) — Cross-Checks Against Independent Method

Monomial e-folds, eq. (25), n = 2 case. $N \approx \varphi^2/(2nM_{Pl}^2)$ with $n = 2$ gives $N \approx \varphi^2/(4M_{Pl}^2)$, i.e. $\varphi_N^2 \approx 4NM_{Pl}^2$ — matching the standard quadratic chaotic-inflation e-fold formula (Linde-type chaotic inflation) as an independent cross-check.

Exponential attractor, eqs. (26)–(27). This cross-check is what caught the original drafting error recorded in B.5.1 below. Redone with the corrected $p = 2/\lambda^2$: $H = p/t$ and $\dot{\varphi} = M_{Pl}\sqrt{(2p)}/t = (2M_{Pl}/\lambda)/t$, so $\varphi^2 = 4M_{Pl}^2/(\lambda^2t^2)$ and $H^2 = p^2/t^2 = 4/(\lambda^4t^2)$. Then, from the equation-of-state definition $1 + w_\varphi = \dot{\varphi}^2/(3M_{Pl}^2H^2)$:

> $1 + w_\varphi = [4M_{Pl}^2/(\lambda^2t^2)] \,/\, [3M_{Pl}^2 \cdot 4/(\lambda^4t^2)] = 4M_{Pl}^2/(\lambda^2t^2) \cdot \lambda^4t^2/(12M_{Pl}^2) = \lambda^2/3$
>
> giving $w_\varphi = -1 + \lambda^2/3$ — the same result as (27), now obtained from the equation-of-state definition directly, and manifestly dimensionless.

Quadratic oscillation, eqs. (28)–(29). Cross-checked via energy bookkeeping: $\rho_\varphi = \tfrac{1}{2}m^2A^2$ (period-averaged) and, independently, $\dot{\rho}_\varphi = -3H\rho_\varphi$ (since $\langle w_\varphi \rangle = 0$) gives $\rho_\varphi \propto a^{-3}$,

hence $A^2 \propto a^{-3}$ and $A \propto a^{-3/2}$, matching (28) without going through the envelope-equation route.

B.5 Correction Record

The verification pass found two genuine errors and two precision issues, all corrected in the main text:

7. **Section 8.2, dimensionally inconsistent attractor exponent (error, corrected).** Equation (26) originally read $p = 2M_{Pl}^2/\lambda^2$. Since $a(t) \propto t^p$ requires p to be a pure number, and λ is already dimensionless, an M_{Pl}^2 could not legitimately survive in p. Retracing the derivation, M_{Pl} cancels completely out of the exponent-matching condition $\lambda\sqrt{(2p)} = 2$, which is a relation between two pure numbers; the correct result is $p = 2/\lambda^2$ and $a(t) \propto t^{2/\lambda^2}$. This error was caught because the B.4 cross-check, redone independently from the equation-of-state definition $1 + w_\varphi = \dot{\varphi}^2/(3M_{Pl}^2H^2)$, produced a non-dimensionless intermediate result when it used the flawed p — the inconsistency was the tell. Equation (27) is unaffected.
8. **Section 9.2, Jeans-mode direction (error, corrected).** As originally drafted, the text asserted that modes with $k < k_J$ are stabilised and modes with $k > k_J$ grow. Testing this against eq. (18) directly shows that for $V''(\varphi_0) > 0$ every mode has $\omega_k^2 = k^2/a^2 + V'' > 0$, so $\delta\varphi$ has no growing mode whatsoever. A genuine instability requires $V''(\varphi_0) < 0$, and in that regime long-wavelength modes ($k < k_J$) grow while short-wavelength modes ($k > k_J$) are gradient-stabilised — the reverse assignment from the original text. The main text now states the corrected, sign-conditional result.
9. **Section 8.1, tensor-to-scalar comparison (precision, corrected).** The original phrasing "current bounds sit roughly an order of magnitude below" the $n = 2$ prediction ($r \approx 0.133$) overstated the gap; current bounds are closer to a factor of 3–4 below, not ~10×. Reworded accordingly.

10. **Section 8.2, exponent notation (cosmetic, corrected).** An intermediate expression carried a redundant M_{Pl}/M_{Pl} left over from an unsimplified exponent; replaced with a one-line explanation of the cancellation.

No other numbered equation or worked result required correction under this pass — in particular, every result in Sections 4–7, 10, and 11, and eqs. (25), (27), (28)–(29) in Section 8, reproduced independently under a second derivation route with no discrepancy.